REDIMAT
Journal of Research in Mathematics Education

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Interpretation of graphical information in PV graphics

Thomas Lingefjärd[1]

1) University of Gothenburg, Sweden

Date of publication: June 24th, 2023
Edition period: June-October 2023

To cite this article: Lingefjärd, T. (2023). Interpretation of graphical information in PV graphics. *REDIMAT – Journal of Research in Mathematics Education*, *12(2)*, 127-143. doi: 10.17583/redimat.11406

To link this article: http://dx.doi.org/10.17583/redimat.11406

REDIMAT, Vol. 12 No. 2 June 2023 pp. 127-143

Interpretation of Graphical Information in PV Graphics

Thomas Lingefjärd
University of Gothenburg

(*Received: 21 November 2022; Accepted: 23 May 2023; Published: 24 June 2023)*

Abstract

Thermodynamic processes are often presented in so called P-V diagrams and the processes are often isobaric, isochoric, isothermal, and adiabatic. The purpose of the qualitative research reported here was to explore students' reasoning and interpretation of P-V diagrams that were presented by the help of GeoGebra. The research group had 15 students and the control group had 12 students. The control group were not given any dynamical explanation of the PV diagrams but were taught with static images in the same text book as the research group. The thermodynamics in Physics 1 at the Swedish gymnasium level is normally done within 2 weeks. We allocated 3 weeks and 10 hours for the thermodynamics to be taught and then gave all 27 students a short test. At the end of the Physics 1 course, we also gave a test with two questions from the thermodynamics. It seems that the students who were given the opportunity to dynamically change the content of the PV diagram benefit more from the teaching compared to the control group.

Keywords: Graphical Information, PV Graphics.

2023 Hipatia Press
ISSN: 2014-3567
DOI: 10.17583/redimat.11406

REDIMAT, Vol. 12 No. 2 June 2023 pp. 127-143

Interpretación de la Información Gráfica en los Gráficos PV

Thomas Lingefjärd
University of Gothenburg

(*Recibido: 21 Noviembre 2022; Aceptado: 23 Mayo 2023; Publicado: 24 Junio 2023)*

Resumen

Los procesos termodinámicos suelen presentarse en los denominados diagramas P-V y los procesos suelen ser isobáricos, isocóricos, isotérmicos y adiabáticos. El objetivo de la investigación cualitativa de la que aquí se informa era explorar el razonamiento y la interpretación de los alumnos de los diagramas P-V que se presentaban con la ayuda de GeoGebra. El grupo de investigación estaba formado por 15 estudiantes y el grupo de control por 12 estudiantes. Al grupo de control no se le dio ninguna explicación dinámica de los diagramas P-V, sino que se les enseñó con imágenes estáticas en el mismo libro de texto que al grupo de investigación. La termodinámica en Física 1 en el nivel de gimnasia sueco se hace normalmente en 2 semanas. Dedicamos 3 semanas y 10 horas a la enseñanza de la termodinámica y, a continuación, sometimos a los 27 estudiantes a un breve examen. Al final del curso de Física 1, también hicimos un examen con dos preguntas de termodinámica. Parece que los estudiantes que tuvieron la oportunidad de cambiar dinámicamente el contenido del diagrama FV se beneficiaron más de la enseñanza en comparación con el grupo de control.

Palabras clave: Información Gráfica, Gráficos PV.

2023 Hipatia Press
ISSN: 2014-3567
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A graph is a powerful tool to interpret data and represent the relationships between variables in various disciplines such as social sciences (psychology, economics, and sociology) and natural sciences (physics, biology, and chemistry). Therefore, the ability to use a graph is an important goal in physics education.

Mathematical representations such as diagrams, histograms, functions, graphs, tables, and symbols facilitate understanding and communication of abstract mathematical concepts or other situations described in mathematical terms (Elby, 2000; Leinhardt, Zaslavsky, & Stein, 1990). Nevertheless, today's humans face a world shaped by increasingly complex, dynamic and powerful information systems through various media. Being able to interpret, understand and work with graphical representations involves mathematical processes the student needs to appreciate, comprehend and be able to address when facing interpretation challenges (Friel, Curcio & Bright, 2001).

For mathematics education in an elementary, middle, lower secondary and upper secondary perspective, teachers use different representations in order to make it possible for students to understand more and more complex mathematical objects and concepts gradually. Geometrical constructions, graphs of functions and various diagrams of different kinds are used to introduce new concepts and study relations, dependency, and change (Trigueros & Martínez-Planell, 2010). Mathematical representations, structures and constructions are also used in different scientific branches, such as biology, chemistry, physics, or social science. It is of major importance that students learn how to interpret graphical representations in a scientific and successful way.

Understanding a graphical representation of a situation requires different concepts be incorporated in the specific representation. The critical problem of transition between and within representations has been addressed in several studies (Breidenbach, Hawks, Nichols & Dubinsky, 1992; Janvier, 1987; Sfard, 1992). They claim that bridging the gap between algebraic and graphic representations depends highly on how students encapsulated relevant concepts involved in the representation.

Since PV diagrams are not common in newspapers or in everyday discussions, PV graphs are difficult for students to relate to anything in their daily life. Heat, volume, internal energy, entropy, pressure, and temperature change in thermodynamics. We can visualise these changes better by making a graphical representation which shows the relationship between these changes and the thermodynamic stages of a process. These graphical representations are known as PV graphs (pressure-volume diagrams). See Figure 1. The constructions in GeoGebra were given in Swedish to the students. I have translated them into English.

A valuable characteristic of PV diagrams and models of thermodynamic processes is their symmetry. One example of this symmetry is an isobaric process (constant pressure) with a volume expansion from state 1 to state 2. You see this in Figure 1.

When it comes to drawing basic PV graphs, the y-axis represents the pressure, and the x-axis represents the volume. Increasing pressure values follow a down-to-up direction, and increasing volume values follow from left to right. An arrow indicates the direction of the processes. The left-to-right direction is positive, while the right-to-left is negative.

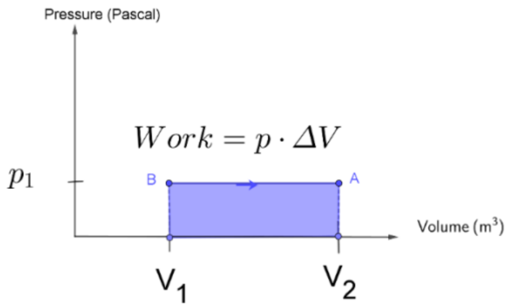

Figure 1. A PV graph of the work in a gas where the area is $p_1 = p \cdot \Delta V$ (the blue region in Figure 1).

Since we have Work (by the gas) defined as $W = p \cdot \Delta V$, when calculating the work of the gas (as pressure per change in volume) in PV diagrams, we can calculate this as the area below the curve. In an isobaric process, the work equals the pressure multiplied by the volume change. If we do not have a graph or if we do not have the formula for a function, we can use the formula instead (see Figure 2).

The Problem

Let us assume that we have an isotherm process with a gas with an amount of 5 moles that expands under constant temperature at 400 K from 3 litres under 6 atm to 9 litres under 2 atm. What work is done by the gas?

If we have a reversible and isothermal process, we use the formula P·ΔV for irreversible expansion at constant pressure. The derivation for the reversible process formula is from the integral:

(1) $$\int_{V_1}^{V_2} P\, dV$$

but if pressure is not constant and we are using the ideal gas law, we use the formula:

(2) $$PV = n \cdot R \cdot T$$

and we substitute P=(nRT)/V so that:

(3) $$\int_{V_1}^{V_2} \frac{nRT}{V}\, dV.$$

Since n, R, and T are constants, we write the integral as:

(4) $$nRT \int_{V_1}^{V_2} \frac{1}{V}\, dV.$$

We now evaluate the integral. Since the $\int \frac{1}{V}\, dV = \ln V$, we get that work by the gas can be written as

(5) $$W = nRT \cdot \ln \frac{V_2}{V_1} \text{ or } W = nRT \cdot \ln \frac{V_{final}}{V_{start}}.$$

This formula was not explicitly explained and derived in the course since most students had not studied integrals before the course in Physics 1.

We have an isotherm process with a gas with an amount of 5 moles that expands under constant temperature at 400 K from 3 litres under 6 atmospheres to 9 litres under 2 atmospheres. What work is done by the gas?

W = n · R · T · ln(VF/V0) = 5 · 8,3145 · 400 · ln(9 litre/3 litre) =18 269 Joule
The ideal gas law can be expressed as P·V = N · k · T where:
P = absolute pressure in atmosphere
V = volume (often in litres)
n = Number of gas particles
k = Boltzmann´s constant (1,38·10−23J·K−1)
T = temperature in Kelvin
The ideal gas law with SI-units is pressure Pascal, volume in m^3, N is n and in moles and k is replaced by R, the gas constant (8,314 J·1/K·1/mol):

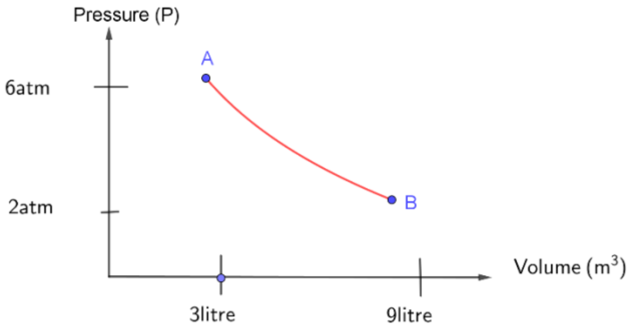

Figure 2. A PV graph of the work in a gas where we do not know the function.

Theoretical Framework

DGE as Amplifier and Reorganizer

Two metaphors are often used in articles about technology in education, for example, that of technology being an 'amplifier' and a 'reorganiser' of mental activity (Pea, 1985, 1987). A DGE therefore plays the role of an 'amplifier' and a 'reorganiser' for developing students' physical thinking. The term 'amplifier' refers to the technology that performs tedious computations (time-consuming to do by hand) quickly and accurately. Therefore, students can focus on making observations and developing insight rather than manual procedures. In this sense, the tool does not change students' thinking but facilitates their explorations.

Nevertheless, when technology is used as a reorganiser, it can extend students' thinking by giving them access to higher-level processes. A DGE supports looking for patterns, identifying invariances or making and testing conjectures. In a paper-pencil environment, students spend significant time drawing and measuring objects. Pea's theory of technology as an amplifier and reorganiser will be used to analyse the teaching of PV graphs and their relation to the GeoGebra applets.

Visualisation and DGE: A Key Aspect in Developing Thinking

A fundamental process in the understanding and construction of physical concepts is visualisation. Allowing the user to drag and manipulate objects, a Dynamic Geometry Environment will facilitate visualisation and conjecture formation. By doing that, it transforms the possibility for representation and has a positive impact on the conceptualisation of objects and internalising their meanings (Falcade, Laborde, & Mariotti, 2007) and (Moreno-Armella, Hegedus, & Kaput, 2008). The contribution of technology in teaching and learning mathematics and physics is perceived as strongly linked with dynamical interactive graphical representations (Laborde, Kynigos, Hollebrands, Strässer, 2006).

In a classroom, various representations, such as diagrams, drawings, and graphs, are used to teach physical concepts. Such multiple representations facilitate and enhance students' understanding of physical concepts. Traditionally, physics was taught and learned in a pencil and paper environment (paper and pencil drawing for constructions) and textbooks providing iconic illustrations. A conceptual understanding of physics requires the development and flexibility of mental imagination. Textbooks with static diagrams cannot highlight figures' dynamic nature over physical situations.

DGE figures and shapes can be manipulated using the dragging feature which provides a dynamic opportunity to the learning of physics. It allows the user to perform investigations and thus affords the possibility of a dynamic visual representation of concepts in a physical sense. Those investigatory activities are hard to experience in a static environment such as paper and pencil (González & Herbst, 2009).

In a DGE, students can experiment with mathematical physics properties and thereby verify conjectures much more easily than in the traditional setting of paper and pencil. The main advantage of a DGE learning environment over other environments is that students use complex figures and easily perform in real time a wide range of transformations on those figures, so students have access to a variety of examples that can hardly be matched by non-computational or static computational environments.

Theory of Variation: Dragging as a Tool in a DGE

Forming mental images is a critical step to abstracting a physical concept. For example, to help students abstract the concept of a pV graphics, the teacher may present stories and perhaps drawings of adiabatic processes, hoping that the student will 'see' some common features among all the figures, the meaning of a pV graphic. In a DGE, the student may drag experience how we could change the value for the area and, through this continuous process, be able to 'see' the properties of the press-volume process, which remain invariant and those that vary. For example, an ideal gas with a volume of 3 or 9 litres changes its pressure if we decrease or increase it. Therefore, the dragging tool is perhaps the most powerful feature of a DGE, as it allows the user to abstract an idea by observing the properties of figures, which remain invariant during the variation process. Leung (2003) aptly describes this affordance of a DGE.

> …..when engaging in learning activities or reasoning, one often tries to comprehend abstract concepts by some kind of "mental animation", i.e. mentally visualising variations of conceptual objects in the hope of "seeing" patterns of variation or invariant properties. ….. one of DGE's power is to equip us with the ability to retain (keep fixed) a background configuration while we can selectively bring to the fore (via dragging) those parts of the whole configuration that interested us in a learning episode.

Over the years, many researchers have studied the role of dragging in DGE, focusing on how it can be instrumental in helping students construct figures using their properties, explore physical problems, formulate conjectures and even proofs. Arzarello, Olivero, Domingo & Ornella (2002), identified seven dragging modalities (wandering, guided, bound, dummy locus, line, linked, drag test) while trying to analyse conjecture-making episodes by students working on a problem.

Variation Theory

Marton, & Booth (1997) proposed four inter-related functions of variation which they also referred to as patterns of variation. These are:

> Contrast: ''… in order to experience something, a person must experience something else to compare it with.''Generalisation: ''… in order to fully understand what ''a graph'' is, we must also experience varying appearances of ''a graph'',…''
> Separation: ''In order to experience a certain aspect of something, and in order to separate this aspect from other aspects, it must vary while other aspects remain invariant.''
> Fusion: ''If there are several critical aspects that the learner has to take into consideration at the same time, they must all be experienced simultaneously.''

They advocated that variation and simultaneity play an important role in the discernment of a concept. According to them, to discern and understand a concept, one must experience variations of it.

Three Aspects of Fidelity

While using technology to explore mathematical concepts and problems, assessing its pedagogical, mathematical, and cognitive fidelity is relevant. Zbiek et al. (2007) describe mathematical fidelity as

> faithfulness of the tool in reflecting the mathematical properties, conventions, and behaviors (as would be understood or expected by the mathematical community) (p.1173).

Let us consider the function $f(x) = (x^2 – 1)/(x – 1)$. Graphics calculators may produce the linear equation $y = x + 1$ if a graphics calculator graphs this function. This is inaccurate since the function $f(x)$ is not defined at $x = 1$, and the correct $f(x)$ graph should have a point break at $x = 1$. Thus, the mathematical fidelity of the tool is compromised in relation to graphing of the function.

Zbeik et al. describes Cognitive fidelity as:

> … the faithfulness of the tool in reflecting the learner's thought processes or strategic choices while engaged in a mathematical activity (Zbiek et al., 2007, p.1173).

A tool has cognitive fidelity if the produced external representations match the user's internal representations and enhance their conceptual understanding. If appropriately used, a DGE has good cognitive fidelity.

The third kind of fidelity is that of Pedagogical fidelity, which, according to Zbeik et al., is:

> … the extent to which teachers (as well as students) believe that a tool allows students to act with a physics concept in ways that correspond to the nature of learning physic that underlies a teacher's practice (Zbiek et al., 2007, p.1187).

Pedagogical fidelity refers to the tool's ability to support students' explorations and learning. In a DGE, the dragging feature can afford this kind of fidelity. We can use sliders to vary the position of a point in a PV graph and observe the change in the graphical representation of an area for the work done by the gas. The level and degree of the types of fidelity vary among technology tools and should be considered while selecting and evaluating appropriate tools for students' explorations. Fidelity must also be considered while designing exploratory tasks for students.

The theoretical frameworks described above may be classified into two categories. The four interrelated functions of variation focus on understanding students' thinking and reasoning in a DGE environment, while Pea's theory of amplifier and reorganiser and the three aspects of fidelity are related to the design and positioning of DGE based geometrical applets for learning physics.

Methodology

The course Physics 1 in a Swedish gymnasium is an overview of many areas of physics. It contains position, velocity, acceleration, forces, electrical fields, pressure, Archimedes principle, Einstein´s postulate, time dilation and relativistic energy, the fundamental forces, energy and energy resources, work, effect, potential and kinetic energy, mechanic, thermic, electric, and chemical energy, radiation and nuclear energy, the energy principle, entropy, degrees of efficiency, the quality of energy, and more. The domain of thermodynamics is often taught rapidly and in a couple of weeks.

There were two gymnasium schools in Sweden where the study took place. In one of the gymnasiums was the research group, and in the other was the group called the control group. Both groups of students studied Physics 1 with the same textbook. The research group had 15 students, and the control group was 12. The control group were not given any dynamic explanation of the PV diagrams. Still, they were taught more traditionally with lectures by one teacher on the blackboard with the static images in the same textbook as the research group.

The thermodynamics was taught for 3 weeks with 10 hours to the research group of students. Every week the students in the research group had a lecture about a thermodynamical situation with a GeoGebra construction to visualise the situation. The students were allowed to explore the construction on the web or in GeoGebra if they had that program on the computer they used at home. All the students had a computer at home. At the end of the thermodynamics, we gave a test, and at the end of the Physics 1 course, we also gave a test with two questions from thermodynamics.

The research explored students' interpretation and reasoning when analysing P-V graphics that were dynamically changeable. See Figure 3.

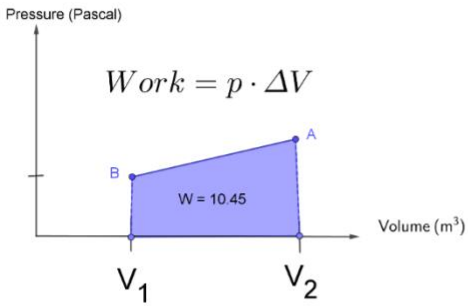

Figure 3. A PV graph of the work in a gas where we can change the position of A, and B (You find the GeoGebra file at https://www.geogebra.org/m/axmp9ghg)

The students were given time to play with the construction. The students were asked questions like "Can you get a work of 12 units?" and "What is the unit for work of a gas?" and "What is the unit along the y-axis?"

An adiabatic process in thermodynamics is a process that appears without transferring heat or mass between the thermodynamic system and its environment. Unlike an isothermal process, an adiabatic process only transfers energy to the surroundings as works. The adiabatic process helps to explain the first law of thermodynamics. Therefore, the students were given the following construction. It introduces the concept of temperature curves and sliders controlling the position of A and B.

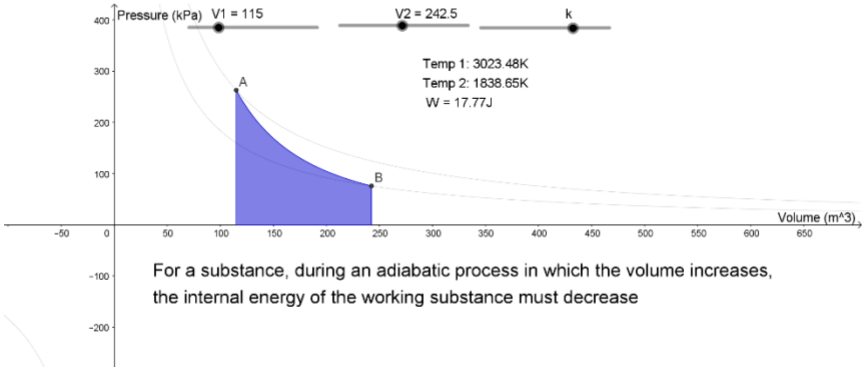

Figure 4. A PV graph of the work in a gas where we see the temperature curves (You find the GeoGebra file at Introducing Temperature Curves – GeoGebra)

In the third week, for the thermodynamic part of Physics 1, the students were given the construction in Figure 5, followed by a discussion about thermodynamical processes for 90 minutes.

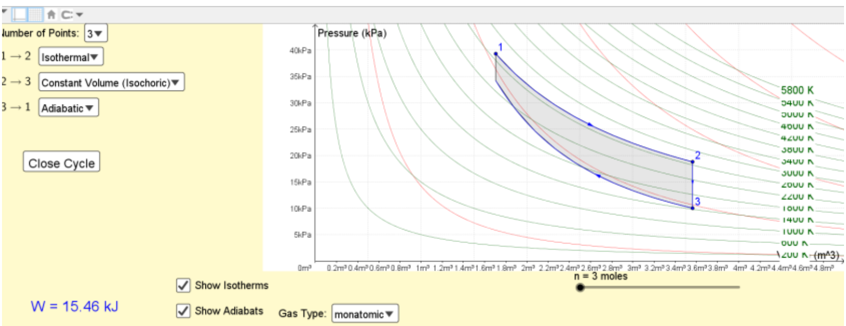

Figure 5. A PV graph of the work in a gas where we can select the isotherms and adiabatic curves (You find the GeoGebra file at "Thermophysics with PV graphics" – GeoGebra)

Results

Altogether, five GeoGebra constructions were distributed and discussed with the 15 students in the research group. Some of the students reacted on the statement written in the GeoGebra construction in Figure 4. For a substance, during an adiabatic process in which the volume increases, the internal energy of the working substance must decrease. One asked me, “If the internal energy decrease, then the substance must be cooler.” This is an interesting question. The substance can still cool down if there is no heat or mass transfer between the thermodynamic system and its environment.

On the last week of the three weeks, we had for the thermodynamic part of Physics 1, the students we given the construction in Figure 5 and the discussion about thermodynamic processes went on for 90 minutes.

At the end of the thermodynamic course, the research group’s 15 students, together with the control group’s 12 students, were given the following question:

> One Argon gas is expanded adiabatically from 0.01 m^3 with the pressure 2 $\cdot 10^5$ Pascal to 0.03 m^3 with the pressure 4 $\cdot 10^5$ Pascal. Calculate the work done by the gas.

Twelve of the 15 students in the research group managed to draw a PV graph, such as the one in Figure 6, while five students in the control group managed to do the same.

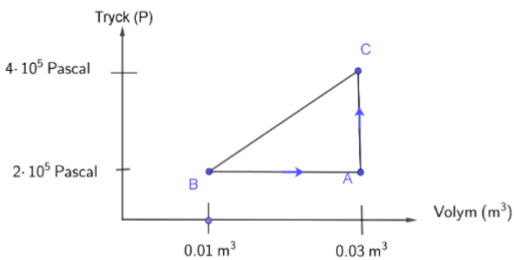

Figure 6. Argon gas is expanded adiabatically from 0.01 m3 with 2 ·105 Pascal to 0.03 m3 with 4 ·105 Pascal.

Ten students in the research group and three in the control group calculated the work done by the gas as the area of the triangle where Area = $0.02 \cdot 2 \cdot 10^5/2$

$= 2 \cdot 10^3$. Two of the students in the control group calculated the work with the formula $W = p \cdot \Delta V = 2 \cdot 10^5 \cdot 0.02 = 4 \cdot 10^3$.

At the end of the Physics 1 course, when the students were waiting for the national exam, we gave a preparation test covering most of the areas in Physics 1. The thermodynamic questions were formulated as follows:

> What work is done in the system below if we go the system as A → B → C → D → A. What is the work done in 50 cycles?

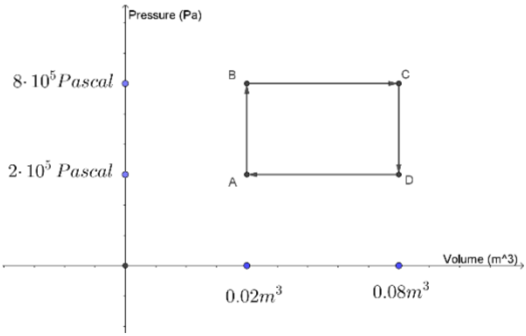

Figure 7. What work is done in the system below if we go the system as A → B → C → D → A?

> Solution (if we go left - right the work is positive while if we go right - left the work is negative.)
> Area = ΔP·ΔV = 6·105 · 0,06m3
> Work = 36 000 Joule
> Look at the different parts of the cycle
> We have that WA → WB = WC→WD = 0
> While WB→WC = 48 000 Joule
> and WD→WA = -12000 Joule.
> Every cycle gives a positive result of 36 000 Joule.
> That gives that 50 cycles are 50 · 36 000 = 1 800 000 J

The other thermodynamic question was to explain the words isobaric, isotherm, and isochoric together with what adiabatic stands for.

Altogether we, the two teachers at that gymnasium, considered it obvious that the dynamic constructions in GeoGebra helped the students to understand the structure of pV graphics.

Discussion

In this study, I was interested in students' interpretations and ability to read the features of the graph, to read between the features of the graph and to read beyond the features of the graph (Friel et al., 2001). Would it help them to get GeoGebra constructions?

The GeoGebra applets for the thermodynamics in the Figure 1 and Figure 2 and the applet in Figure 3 form a ground for the physics we use later in the article. In this regard, these applets may be considered to have a high level of physical fidelity.

The applets also had a degree of cognitive fidelity since they assist our thought processes while we engage in the physical thinking of pV graphics. The applets, with their physical and cognitive fidelity, create an environment which enables us to use them to make a conjecture about ideal gas in a process. Finally, the applets also had a high level of pedagogical fidelity as they help us to further our exploration and learning. The applets encourage the participation of the readers. The fidelity of technological tools is an important criterion while evaluating their use in the classroom for physical exploration and learning.

When visualising pV graphics, students are in a new position. The coordinate system should be interpreted before analysing the geometry in the pV graphics. In the GeoGebra construction, you can vary the area of the work by moving points. By doing this in a teaching sequence or for students doing it by themselves, they will learn through the variation and amplify and reorganise their conceptual understanding.

GeoGebra enabled applets may very well function as amplifiers and reorganisers in enabling students to explore the fundamental concepts related to pV graphics. The essential characteristic (of the GeoGebra applets) is that the position of a point is something completely else on the screen teachers must keep in mind while preparing tasks that function as amplifiers or reorganisers in enabling students' understanding of concepts of pV graphics. Teachers should use the applets to introduce and discuss the concepts of pV graphics, scaffolding students' explorations and helping them develop the concepts.

The three cognitive conditions in the framework of Friel et al. (2001)—to read the features of the graph, to read between the features of the graph and to read beyond the features of the graph—have been identified amongst student's responses. My findings, based on this and previous studies, indicate that discussing and teaching students about scientific concepts may help them learn and understand the mathematical construction of science processes such as pV graphics. Helping students understand the challenging topics in science and mathematics through ontology training may facilitate the learning process.

For the students in the research group, we also see evidence of DGE acting as an amplifier. Dragging points A and point B, in Figure 3, enabled the students to see how the work of the gas was recalculated quickly. Thus, the dragging feature led them to visualise their 'mental calculation' quickly and efficiently. As an amplifier, the applet enabled the students to visualise the areas as the work by the gas. In this process, the applet played the role of a reorganiser. Here the focus of students' thinking shifted from viewing the pV graphics as a graph over to viewing it as a work by a gas. Hence, I found ample evidence of the GeoGebra applets playing the role of amplifier and reorganiser in the tasks with pV graphics.

The dragging feature of a DGE enables students to explore, experiment, observe variations and invariances, and verify the permanence or lack of permanence of mathematical properties, leading to conjecture-making more easily than in other digital environments where dragging is impossible. Such dynamism is missing in the more traditional paper and pencil environment. In a DGE learning environment, students can use constructions of a pV graphics and change it until the hidden relations are obvious to the viewer. Such an affordance can hardly be matched by non-computational or static computational environments (Marrades & Gutiérrez, 2000, p. 96). In this study, the students in the research group used the dragging feature of the GeoGebra applet to try various possibilities of adjusting the geometrical shapes representing the area of the pV graphics. This enabled them to explore different positions quickly, a fact which supports Pea's (1987) metaphor of technology as an amplifier. A DGE like GeoGebra can provide students access to multiple solutions in a short span of time.

Limitations

This is just a small case study, not anything that can be used as an argument. This is just a new way to teach physics.

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Thomas Lingefjärd is associate professor of Mathematics Education in the Faculty of Education, at the University of Gothenburg, Sweden.

Contact Address: Direct correspondence concerning this article should be addressed to the author. **Postal Address:** Västra Hamngatan 25 41117 Göteborg. **Email:** thomas.lingefjard@gu.se